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On the Convergence of Entropy for Stationary Exclusion Processes with Open Boundaries

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We prove that, for a wide class of stochastic lattice gases in contact with reservoirs, despite long-range correlations, the leading-order term of the Gibbs–Shannon entropy in the nonequilibrium stationary state is given by the local equilibrium entropy.

KEY WORDS: exclusion process, open boundaries, nonequilibrium stationary state, hydrostatic limit, entropy, local equilibrium

1. INTRODUCTION

Stochastic lattice gases in contact with reservoirs provide typical examples of nonequilibrium stationary states (NSS). These models are markov processes on a finite lattice of size N, on which particles hop with some interaction rules, and may be created or killed at the edges to model interaction with reservoirs. The study of large-scale properties (i.e. $N \to \infty$) of such systems is a subject of sustained interest in statistical physics. Particle number is the only conserved quantity in the bulk, hence the system is described at macroscopic level by a single density field $\rho(x)$, $x \in (0, 1)$. Reservoirs are characterized by their fixed densities ρ_l, ρ_r. For the class of systems we investigate, the evolution of the typical density field is given to the leading order by a drift-diffusion equation (also called *hydrodynamic limit,* see Ref. 14):

$$\partial_t \rho + \partial_x f(\rho) = \partial_x[D(\rho)\partial_x \rho] \tag{1}$$

For purely diffusive systems such as the symmetric exclusion process (SSEP) the current $f(\rho)$ vanishes; current and diffusion terms are both involved for

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weakly driven diffusive systems such as the weakly asymmetric exclusion process (WASEP); finally the diffusion term is absent in the case of strongly driven systems like the asymmetric exclusion process (ASEP), in which case solutions are understood in weak entropy sense and exhibit shocks.[19,20] Microscopically, the nonequilibrium steady state μ^N of the open system is a probability measure on the set of all lattice configurations. It is described macroscopically by the typical stationary density field $\rho_s(x)$, $x \in (0, 1)$, which is the stationary solution to (1) with Dirichlet boundary conditions ρ_l, ρ_r. For diffusive systems, this was established e.g. in Ref. 12 for lattice gases with gradient reversible bulk dynamics, and in Ref. 15 for nongradient reversible bulk dynamics. For the asymmetric exclusion process, the stationary profile was derived in Ref. 5 (see also Ref. 17) and extended in Refs. 1 and 18 to more general driven systems; in these cases the stationary profile is found to be uniquely defined only outside a boundary-induced phase transition line in the (ρ_l, ρ_r) plane, and boundary conditions must be interpreted in a special sense[1,3] that permits boundary shocks.

Fluctuations from the typical profile in the NSS reveal long-range correlations, a major difference compared to equilibrium states of infinite systems or systems with periodic boundaries. For the open SSEP,[21] fluctuations around $\rho_s(.)$ are gaussian of order $N^{-1/2}$, with a covariance of the form

$$C(x, y) = \chi(\rho_s(x))\delta(x - y) - (\rho_l - \rho_r)^2(-\Delta)^{-1}(x, y) \tag{2}$$

where $(-\Delta)^{-1}(x, y) = (x \wedge y)(1 - x \vee y)$ is the inverse (minus) Laplacian with 0 Dirichlet boundary conditions. While the first (white-noise) term reflects local equilibrium, where $\chi(\rho)$ is the static compressibility, the second term reflects long-range correlations. A similar decomposition holds for WASEP[7] and more general lattice gases[12,22]; for the asymmetric exclusion process, the order is still $N^{-1/2}$, but the non-local component has been shown[6] to be non-gaussian. Long-range correlations also appear in the study of large fluctuations from $\rho_s(\cdot)$. These are described by the *large deviation* functional $\mathcal{F}[\rho(\cdot)]$ which gives the asymtoptic probability of an untypical profile $\rho(\cdot)$ in the NSS: in loose style,

$$\mu^N[\rho(\cdot)] \sim e^{-N\mathcal{F}[\rho(\cdot)]} \tag{3}$$

with $\mathcal{F}[\rho_s(\cdot)] = 0, \mathcal{F}[\rho(\cdot)] > 0$ for $\rho(\cdot) \neq \rho_s(\cdot)$. Unlike in equilibrium states, where $\mathcal{F}$ has a local form

$$\mathcal{F}[\rho(\cdot)] = \int_0^1 s(\rho(x))\, dx \tag{4}$$

this functional has been shown to be nonlocal for some diffusive or driven systems with open boundaries (see e.g. Refs. 2, 4, 8, 9).

Another natural quantity in which long-range effects may be investigated is Gibbs–Shannon entropy,

$$S(\mu^N) := -\sum_{\eta \in X_N} \mu^N(\eta) \log \mu^N(\eta)$$

where summation is over all lattice configurations η. For an equilibrium Gibbs state with hamiltonian H at density ρ, it is well-known[11] that the specific entropy $N^{-1} S(\mu^N)$ converges to some value $s_H(\rho)$, for instance we have

$$s_H(\rho) = -[\rho \ln(\rho) + (1-\rho)\ln(1-\rho)]$$

in the case of SSEP, where $H \equiv 0$, hard core being the only interaction. For a Gibbs state with slowly varying chemical potential corresponding to the profile $\rho(\cdot)$, one easily obtains convergence of the specific entropy to the corresponding local functional

$$S_H^{\mathrm{loc}}[\rho(\cdot)] = \int_0^1 s_H(\rho(x))\, dx$$

A natural question is whether, in the spirit of (2), the specific entropy of the NSS has an additional non-local term to the local equilibrium functional S_H^{loc}. This problem is addressed here and in Ref. 10 in complementary fashions. Sharp analytic computations and combinatorics are used in Ref. 10 for the SSEP to obtain a formal proof of the following result: local equilibrium entropy is the only leading order term, while an explicit nonlocal correction is computed. This correction is of order N^{-1} as opposed to (2), and depends in an explicit way on the non-local correction in the covariance (2). Higher-order corrections are presumably accessible by that method too, and numerical evidence is given for more general systems. In this paper we give a rigorous proof that the leading order term is indeed given by the local equilibrium entropy. We use a rather general argument that is valid for many different systems, driven or not, but gives no information on the next-order corrections. Our argument is based on a result of Kosygina,[16] which states that local equilibrium (in an adequate sense) is essentially sufficient to deal with the leading-order asymptotics of entropy. To avoid heavy notations and computations we restrict detailed proofs to the (symmetric or asymmetric) exclusion process. We then define a fairly general framework to which the arguments extend, and briefly explain why they do extend. This setting includes systems whose conservative part of the dynamics is stationary (but not necessarily reversible) with respect to finite-range Gibbs measures.

2. EXCLUSION PROCESS WITH OPEN BOUNDARIES

We consider the exclusion process on the finite lattice $L_N = \{1, \ldots, N\}$ coupled with particle reservoirs at both ends. The state variable is $\eta \in X_N :=$

$\{0, 1\}^{L_N}$, where $\eta(x) \in \{0, 1\}$ is the number of particles at site $x \in L_N$. The left (right) reservoir density is $\rho_l \in [0, 1]$ ($\rho_r \in [0, 1]$). The dynamics is defined by the Markov infinitesimal generator

$$\begin{aligned}\mathcal{L}^N_{\text{op}} f(\eta) = &\sum_{x=1}^{N-1} p\eta(x)(1 - \eta(x+1))[f(\eta^{x,x+1}) - f(\eta)] \\ &+ \sum_{x=2}^{N} q\eta(x)(1 - \eta(x-1))[f(\eta^{x,x-1}) - f(\eta)] \\ &+ \alpha(1 - \eta(1))[f(\eta + \delta_1) - f(\eta)] + \beta\eta(1)[f(\eta - \delta_1) - f(\eta)] \\ &+ \alpha'(1 - \eta(N))[f(\eta + \delta_N) - f(\eta)] + \beta'\eta(N)[f(\eta - \delta_N) - f(\eta)]\end{aligned} \tag{5}$$

where, for $x, y \in \mathbb{Z}$, $\eta^{x,y}$ (resp. $\eta + \delta_x$, $\eta - \delta_x$) denotes the new particle configuration after a particle has jumped from x to y (resp. has been created at x, killed at x). The birth and death rates $\alpha, \beta, \alpha', \beta'$ are such that

$$\frac{\alpha}{\rho_l} - \frac{\beta}{1 - \rho_l} = p - q, \qquad \frac{\alpha'}{\rho_r} - \frac{\beta'}{1 - \rho_r} = q - p \tag{6}$$

for which a natural choice is

$$\alpha = p\rho_l, \quad \beta = q(1 - \rho_l), \qquad \alpha' = q\rho_r, \quad \beta' = p(1 - \rho_r) \tag{7}$$

The Markov process defined by (5) has a unique invariant measure (or steady state) μ^N on X_N. The existence of an explicit limiting density profile (or hydrostatic limit) $\rho_s(\cdot)$ for μ^N was established in previous works: see e.g. Ref. 12 for the symmetric case $p = q = 1/2$, Refs. 5 and 17 for the asymmetric case ($p \neq 1/2$). These two cases are qualitatively different:

Symmetric case. Here

$$\rho_s(x) = \rho_l(1 - x) + \rho_r x$$

is the stationary solution to the hydrodynamic equation

$$\partial_t \rho = \frac{1}{2}\Delta\rho \tag{8}$$

in (0; 1) with boundary conditions

$$\rho(t, 0^+) = \rho_l, \qquad \rho(t, 1^-) = \rho_r \tag{9}$$

Asymmetric case. We may assume without restriction that the mean drift $\gamma = p - q > 0$. The stationary profile is uniform:

$$\rho_s(x) \equiv R(\rho_l, \rho_r)$$

where

$$R(\rho_l, \rho_r) = \begin{cases} 1/2 & \text{if } \rho_l \geq 1/2 \text{ and } \rho_r \leq 1/2 \\ \rho_r & \text{if } \rho_r \geq 1/2 \text{ and } \rho_l + \rho_r > 1 \\ \rho_l & \text{if } \rho_l \leq 1/2 \text{ and } \rho_l + \rho_r < 1 \end{cases} \tag{10}$$

In the case $\rho_l < \rho_r$, $\rho_l + \rho_r = 1$ (phase transition line), there is no definite profile, but a uniformly located shock connecting ρ_l and ρ_r. The hydrodynamic equation (see e.g. Ref. 19) is now a viscousless conservation law:

$$\partial_t \rho + \partial_x[\gamma\rho(1-\rho)] = 0 \tag{11}$$

whose solutions are taken in weak entropy sense and generally exhibit shocks. In this case, since the current-density function is strictly concave, only upward shocks are possible. A striking difference with the diffusive case is that there cannot be a stationary solution with boundary data $\rho_l \neq \rho_r$ in a usual sense if $\rho_l + \rho_r \neq 1$: indeed, (11) would imply that such a solution is constant or consists of a single upward shock $\rho^- < \rho^+$, with $\rho^- + \rho^+ = 1$ by flux continuity. However it has been observed ([1]) that $\rho_s(\cdot)$ could still be interpreted as the unique stationary solution to (11) with boundary conditions ρ_l, ρ_r, if these are taken in the "BLN" sense,(3) which allows boundary shocks. On the phase transition line the set of stationary solutions (in this case, both in BLN or usual sense with respect to boundaries) consists of arbitrarily located shocks connecting ρ_l and ρ_r.

3. CONVERGENCE OF ENTROPY

We consider the Gibbs–Shannon entropy of μ^N, defined by

$$S(\mu^N) := -\sum_{\eta \in X_N} \mu^N(\eta) \log \mu^N(\eta)$$

For the translation-invariant exclusion process on the finite lattice $\{1, \ldots, N\}$ with periodic boundary conditions, equilibrium states are given by the product Bernoulli measures ν_ρ^N with densities $\rho \in [0, 1]$ (in the sequel we shall denote the finite lattice by T_N rather than L_N when periodic boundary conditions are used). The corresponding equilibrium entropy is given by

$$s(\rho) = -[\rho \ln \rho + (1-\rho) \ln], \qquad \rho \in [0; 1]$$

since an explicit computation yields $S(\nu_\rho^N) = Ns(\rho)$. More generally, if ν^N is product with mean $\rho^N(x)$ at site x, and $\rho^N([Nx])$ converges to some limiting profile $\rho(x)$ in $L^1((0,1))$ as $N \to \infty$ (where [ ] denotes integer part), it is easy to see that

$$\lim_{N\to\infty} N^{-1} S(\nu^N) = \mathcal{S}[\rho(\cdot)] := \int_0^1 s(\rho(x))\, dx$$

The nonequilibrium steady state μ^N is close to such a measure only locally, but our main result states that long-range correlations do not affect the leading order term of entropy:

Theorem 3.1. *In the symmetric case, or in the asymmetric case in any of the cases (10), we have* $N^{-1}S(\mu^N) \to \mathcal{S}(\rho_s(\cdot))$ *as* $N \to \infty$.

The proof of Theorem 3.1 is based on a result of Kosygina (Lemma A.1. of Ref. 16), which shows that local equilibrium (in a strong enough sense controlled by the Dirichlet form) is enough to ensure convergence of the specific entropy to the local equilibrium entropy. Let us denote by D^N the Dirichlet form for the symmetric exclusion process on L_N:

$$D^N(\mu) = \frac{1}{2} \sum_{x,y \in L_N : x \sim y} \sum_{\eta} \left[\sqrt{\mu(\eta^{x,y})} - \sqrt{\mu(\eta)}\right]^2$$

$$= -\int_{X_N} \sqrt{f(\eta)} \mathcal{L}_s^N \sqrt{f(\eta)} \nu_\rho^N(d\eta) \tag{12}$$

where μ is a probability measure on X_N. On the first line of (12), $x \sim y$ means that x, y are neighbouring sites. On the second line $\rho \in (0, 1)$, $f = d\mu/d\nu_\rho^N$, and $\mathcal{L}_s^N$ is the generator of the symmetric exclusion process on L_N:

$$\mathcal{L}_s^N f(\eta) = \sum_{x=1}^{N-1} \frac{1}{2}\eta(x)(1-\eta(x+1))[f(\eta^{x,x+1}) - f(\eta)]$$

$$+ \sum_{x=2}^{N} \frac{1}{2}\eta(x)(1-\eta(x-1))[f(\eta^{x,x-1}) - f(\eta)]$$

The second line of (12) is independent of the choice of ρ. $D^N(\mu)$ vanishes iff μ is a combination of Bernoulli measures. To state Kosygina's result we use the standard notation

$$\eta^l(x) := (2l+1)^{-1} \sum_{y \in T_N : |y-x| \le l} \eta(y)$$

to denote the empirical particle density in a block of radius l centered at x.

Proposition 3.1.[16] *Assume* (μ^N) *is a sequence of probability measures on* X_N *such that:*

i) μ^N *satisfies the local equilibrium bound*

$$\lim_{N\to\infty} N^{-1} D^N(\mu^N) = 0 \tag{13}$$

ii) μ^N *has some limiting density profile* $\rho(\cdot)$ *in the sense*

$$\lim_{l\to\infty} \limsup_{N\to\infty} \mathbb{E}_{\mu^N} \left\{ \int_0^1 |\eta^l([Nx]) - \rho(x)| dx \right\} = 0 \tag{14}$$

Then

$$\lim_{N\to\infty} N^{-1} S(\mu^N) = \mathcal{S}[\rho(\cdot)] \tag{15}$$

Remark 1. Assumption (ii) is stronger than mere existence of a hydrodynamic profile $\rho(\cdot)$ in the usual sense, as the latter means that the limiting profile is achieved by small macroscopic blocks (i.e. of size $N\varepsilon$ with $\varepsilon \to 0$ after $N \to \infty$), whereas (14) states that the profile is achieved by large microscopic blocks (i.e. of size $l \to \infty$ after $N \to \infty$). In fact (14) is equivalent to existence of a hydrodynamic profile in the usual sense (Assumption (Al) in Lemma A.1 of Ref. 16) *plus* a two-block estimate (Assumption (A3) in the same lemma), which says that large microscopic blocks are close to small macroscopic blocks. It is immediate that (A1)–(A3) of Ref. 16 imply (14). Conversely, (14) easily implies (A1) of [Ref. 16], because a small macroscopic block can be decomposed into large microscopic blocks; next, (14) and (A1) of Ref. 16 immediately imply (A3).

Remark 2. For diffusive systems, the two-block estimate is generally obtained a *priori* as a result of a more refined bound of order $O(1/N)$ on the Dirichlet form (see also remark at the end of Sec. 4.3). For asymmetric systems one cannot get this refined bound and thus the a priori two-block estimate, but the method of proof directly yields the hydrodynamic or hydrostatic limit in the strong form (14). See e.g. Ref. 14 for a complete overview of these differences.

Given Proposition 3.1, the proof of Theorem 3.1 is quite simple. By available results,[5,12,17] Assumption (i) of Proposition 3.1 is satisfied with $\rho = \rho_s(\cdot)$, the stationary solution of the hydrodynamic equation with boundary conditions. All we have to verify is (13).

Proof of the bound (13). Let us denote *relative* entropy of two probability measures μ, ν on X_N by

$$S(\mu|\nu) := \int_{X_N} \ln \frac{d\mu}{d\nu} d\mu$$

if $\mu \ll \nu$ and the integral is well-defined, $+\infty$ otherwise. Let $\nu^N = \nu_\rho^N$ for a fixed $\rho \in (0, 1)$. By a standard computation (see e.g. Ref. 14),

$$\frac{d}{dt} S(\mu_t|\nu^N) \le 2 \int_{X_N} \sqrt{f_t(\eta)} \mathcal{L}_{\rm op}^N \sqrt{f_t(\eta)} \nu^N(d\eta) \tag{16}$$

where $\mathcal{L}_{\text{op}}^N$ is the generator (5) of the exclusion process with open boundaries, μ_t is the distribution of this process at time t, and $f_t = d\mu_t/d\nu^N$. The key point is the following estimate for a probability measure $d\mu = f d\nu^N$ on X_N:

$$\begin{aligned}\int_{X_N} \sqrt{f(\eta)} L_{\text{op}}^N \sqrt{f(\eta)} \nu^N(d\eta) &\leq \int_{X_N} \sqrt{f(\eta)} L_s^N \sqrt{f(\eta)} \nu^N(d\eta) \\ &\quad + \int_{X_N} V_N(\eta) f(\eta) \nu^N(d\eta) \\ &= -D^N(\mu) + \int_{X_N} V_N(\eta) f(\eta) \nu^N(d\eta) \quad (17)\end{aligned}$$

where

$$\begin{aligned}V_N(\eta) = \frac{1}{2}\left(\frac{\alpha}{\rho} - \frac{\beta}{1-\rho}\right)(\eta(1) - \rho) + \frac{1}{2}\left(\frac{\alpha'}{\rho} - \frac{\beta'}{1-\rho}\right)(\eta(N) - \rho) \\ + \frac{1}{2}(p - q)(\eta(N) - \eta(1)) \quad (18)\end{aligned}$$

To establish (17) one observes that, by standard computations,

$$\mathcal{L}_{x,x+1}^* = \mathcal{L}_{x+1,x} + (\eta(x+1) - \eta(x)) \quad (19)$$

$$\mathcal{L}_{x+1,x}^* = \mathcal{L}_{x,x+1} + (\eta(x) - \eta(x+1)) \quad (20)$$

$$\mathcal{L}_{x+}^* = \frac{1-\rho}{\rho}\mathcal{L}_{x-} + \frac{\eta(x) - \rho}{\rho} \quad (21)$$

$$\mathcal{L}_{x-}^* = \frac{\rho}{1-\rho}\mathcal{L}_{x+} + \frac{\rho - \eta(x)}{1-\rho} \quad (22)$$

In the above equations $\mathcal{L}_{x,y}$, resp. $\mathcal{L}_{x+}$, $\mathcal{L}_{x-}$ denote the piece of (5) corresponding to a rate 1 jump from x to y, resp. particle creation at x, particle annihilation at x. $*$ denotes adjoint w.r.t. ν^N, and the functions on the r.h.s. denote multiplication operators. We use the fact that

$$\int \sqrt{f(\eta)} \mathcal{L}_{x,y} \sqrt{f(\eta)} d\nu^N = \int \sqrt{f(\eta)} \frac{\mathcal{L}_{x,y} + \mathcal{L}_{x,y}^*}{2} \sqrt{f(\eta)} d\nu^N$$

so that lattice summation of (19)–(20) yields $-D_N(\mu)$ and the third term on the r.h.s. of (18). For the boundary terms we write

$$\begin{aligned}\int \sqrt{f(\eta)} \mathcal{L}_{x,\pm} \sqrt{f(\eta)} d\nu^N &= \int \left\{\sqrt{f(\eta)} \mathcal{L}_{x,\pm} \sqrt{f(\eta)} - \frac{1}{2}\mathcal{L}_{x,\pm} f(\eta)\right\} d\nu^N \\ &\quad + \frac{1}{2}\int \mathcal{L}_{x,\pm} f(\eta) d\nu^N \leq \frac{1}{2}\int f(\eta) \mathcal{L}_{x,\pm}^* 1(\eta)\, d\nu^N\end{aligned}$$

where we used the fact that $\mathcal{L}g^2 - 2g\mathcal{L}g \geq 0$ for any Markov generator $\mathcal{L}$ and test function g. We then use (21)–(22), which yields the first two terms on the r.h.s. of (18).

Taking $\mu_0 = \mu^N$ in (16), we still have $\mu_t = \mu^N$ for all $t > 0$, since μ^N is stationary for $\mathcal{L}_{\text{op}}^N$. Thus we have 0 on the l.h.s. of (16), and (17) yields

$$D^N(\mu^N) \leq \int_{X_N} f_t(\eta) V(\eta) \nu^N dn = \mathbb{E}_{\mu^N}[V(\eta)]$$

By (18), V_N is a uniformly bounded function. Hence $D^N(\mu^N) = O(1)$, which establishes the bound (13).

4. MORE GENERAL MODELS

The main ingredients in the proof of Theorem 3.1 can be summarized as follows.

0) We know (by some former result) that the open system has a limiting profile $\rho_s(\cdot)$ in the sense (14).
1) We have a family of stationary measures (indexed by mean density) ν_ρ^N for the conservative dynamics on the torus with periodic boundary conditions.
2) There is an associated Dirichlet form $D^N(\mu)$ which vanishes only for combinations of ν_ρ^N. A control of order $o(N)$ on $D^N(\mu)$ means that locally μ is close to such a combination.
3) For the stationary system with open boundaries, a $O(1)$ control on D^N follows from the fact that the open system only differs from the periodic system by finitely many bounded boundary terms, this is why we have (17).
4) We have a result (Proposition 3.1) stating that $o(N)$ control for $D^N(\mu)$ form is enough to ensure that the specific entropy converges to the local equilibrium entropy corresponding to the limiting profile.

These ingredients are not specific to the nearest-neighbor exclusion process.

4.1. The Setting

Let us sketch a fairly general framework in which 1)–4) are valid. Derivation of the hydrostatic limit in the sense (14) may require additional conditions, but we are not concerned here with this step, as we want to describe a framework in which Theorem 3.1 simply follows from the hydrostatic limit.

We may consider systems with at most K particles per site, $K \geq 1$. The conservative bulk dynamics is governed by local, finite-range, translation-invariant jump rates $c(x, y, \eta) = c(y - x\tau_x\eta)$, where τ_x denotes space shift. For given (large enough) torus size $l \in \mathbb{N}$, we may define the conservative dynamics with periodic boundary conditions, that is the Markov process with generator

$$\mathcal{L}^l_{\text{per}} f(\eta) = \sum_{x,y \in T_l} c(x, y, \eta)[f(\eta^{x,y}) - f(\eta)] \tag{23}$$

where, in $c(x, y, \eta) = c(y - x, \tau_x \eta)$, $y - x$ is intepreted in periodic sense (e.g. $l - 1 = -1$, etc.). The key assumption is the

Stationarity condition. *There exists a finite-range hamiltonian H such that, for large enough $l \in \mathbb{N}$, the markov process (23) is (a) stationary and (b) ergodic under the canonical Gibbs measures with fixed particle number*:

$$V_H^{l,k}(\eta) := \left(Z_{l,k}^H\right)^{-1} \exp\left[H^l_{\text{per}}(\eta)\right] \quad \text{on} \quad \left\{\sum_{x \in T} \eta(x) = k\right\}$$

where H^l_{per} denotes the total hamiltonian on T_l with periodic boundary conditions.

Note that we do *not* need reversibility, i.e. detailed balance conditions, which do hold for SSEP but not for ASEP. For the open system we add local boundary dynamics governed by Markov generators $\mathcal{A}$ and $\mathcal{B}$, so that the generator of the open system is

$$\mathcal{L}^N_{\text{op}} f(\eta) = \sum_{x,y \in L_N} c(x, y, \eta)[f(\eta^{x,y}) - f(\eta)] + \mathcal{A} f(\eta) + \tau_N \mathcal{B}_{\tau_{-N}} f(\eta) \tag{24}$$

The space shifts around $\mathcal{B}$ mean that $\mathcal{B}$ is a fixed generator that we center around the right boundary. For the purpose of 3) it is enough to know that $\mathcal{A}$ and $\mathcal{B}$ are Markov generators acting on finitely many sites. For x, y near the boundaries $c(x, y, \eta)$ may involve outer sites, in this case we may take given boundary conditions or omit such $c(x, y, \eta)$.

4.2. The Dirichlet Form and Specific Entropy

We may define a Dirichlet form with respect to the hamiltonian H, namely

$$D^N(\mu) = \sum_{x,y} \sum_{\eta \in X_N} \frac{c(x, y, \eta) + c^*(x, y, \eta)}{2} \left(\sqrt{\mu(\eta^{x,y}) e^{-\Delta_{x,y} H(\eta)}} - \sqrt{\mu(\eta)}\right)^2 \tag{25}$$

where

$$\Delta_{x,y} H(\eta) = H(\eta^{x,y}) - H(\eta)$$

$c^*(x, y, \eta)$ are the adjoint rates given by

$$c^*(x, y, \eta) = c(y, x, \eta^{x,y}) e^{\Delta_{x,y} H(\eta)} \tag{26}$$

We claim that the $O(1)$ bound for $D^N(\mu^N)$ is still valid (regardless of the choice of boundary conditions in (25)). Indeed, simple computations show that (19)–(20) extend as follows:

$$\left(\mathcal{L}^c_{x,y}\right)^* = \mathcal{L}^{c^*}_{y,x} + c^*(y,x,\eta) - c(x,y,\eta) \tag{27}$$

where $*$ means adjoint w.r.t. the Gibbs measure ν^N with arbitrary chemical potential λ, and $\mathcal{L}^c_{x,y}$ is the piece of generator corresponding to a jump from x to y with rate c. Likewise, computations like (21)–(22) show that $\mathcal{A}^*1$ and $\mathcal{B}^*1$ are uniformly bounded functions depending on finitely many sites around the boundaries. It is easy to understand how the derivation of (17) and (18) from (19)–(20) and (21)–(22) can be generalized here: the stationarity condition on $c(x,y,\eta)$ exactly means that summation of $c^*(y,x,\eta) - c(x,y,\eta)$ over the torus vanishes. For the open system, summation is not on T_N but on L_N, we are then left with a remainder of the form $F_l(\eta) + F_r(\eta)$, where $F_{l/r}$ are uniformly bounded functions localized within finite distance of the boundaries: in the case of SEP we had $F_l(\eta) = \frac{1}{2}(p-q)\eta(N)$ and $F_r(\eta) = \frac{1}{2}(q-p)\eta(1)$ in (18).

Next, the proof of Proposition 3.1 in Ref. 16 can be refined (using large deviation estimates for Gibbs measures) to establish that $o(N)$ for the more general Dirichlet form (25) implies (15), with a new equilibrium entropy $s(\rho)$ relative to the hamiltonian H. Precisely, let ν_ρ denote the infinite-volume Gibbs measure with density ρ, and ν^N_ρ its projection on the finite lattice L_N. Then the specific entropy is defined by

$$s(\rho) := \lim_{N\to\infty} N^{-1} S\left(\nu^N_\rho\right)$$

(see e.g. Ref. 11 for existence of the limit), and (15) holds with this $s(\rho)$.

4.3. Summary

All the above is independent of whether the leading-order hydrodynamics (1) associated with the rates $c(x,y,\eta)$ is a diffusion equation (as in SSEP), or a viscousless conservation law (as in ASEP). To get a full drift-diffusion equation (1) we must add to (24) (as in WASEP) a weakly asymmetric part governed by local jump rates $N^{-1}c'(x,y,\eta)$. The bound (13) will still be satisfied: indeed, repeating the arguments of Sec. 4.2, we find that the overall contribution of the weakly asymmetric perturbation $N^{-1}c'$ to $D^N(\mu^N)$ is at most $O(1)$.

In summary, assume we have a strongly asymmetric, or symmetric, or weakly asymmetric bulk dynamics as described above, with adequate local boundary dynamics, so that we know the steady state of the open system has a hydrostatic limit in the sense (14). This profile is the unique stationary solution to (1), with the boundary conditions (9). Then we automatically have (15) as a consequence of the hydrostatic limit, Dirichlet form bound and extended Kosygina's result, and this does not use any specific features of the dynamics other than those mentioned above (local interactions, finite-range jumps and stationarity condition). Situations where (14) has been proved include the following: reversible gradient dynamics wrt Gibbs measures[12] and reversible nongradient K-exclusion process[15] in the purely diffusive case; TASEP[5,17] and more general attractive asymmetric particle systems with product invariant measures[1] in the pure drift case. We believe (see remark below) that the result of Ref. 12 extends in the presence of a weakly asymmetric component.

As in the particular case of simple exclusion, uniqueness of the stationary solution always holds in the diffusive case (provided $D(\rho) > 0$ for all ρ).

In the pure drift case, boundary conditions are assumed in the special BLN sense: $\rho(t, 0^+) = \rho_l$ is replaced with $\rho(t, 0^+) \in \mathcal{E}(\rho_l)$ and $\rho(t, 1^-) = \rho_l$ is replaced with $\rho(t, 1^-) \in \mathcal{E}(\rho_r)$, where $\mathcal{E}(\rho_{l/r})$ are generally not reduced to singletons (see Refs. 3 or 1). Uniqueness of the stationary solution occurs iff[1] (ρ_l, ρ_r) is such that

$$\begin{aligned} &\min_{[\rho_l,\rho_r]} f \text{ is uniquely achieved if } \rho_l < \rho_r \\ &\max_{[\rho_r,\rho_l]} f \text{ is uniquely achieved if } \rho_l > \rho_r \end{aligned} \tag{28}$$

in which case the stationary solution is the uniform profile with density $R(\rho_l, \rho_r)$, the unique minimizer or maximizer in (28). The complement of (28) is a union of phase transition lines, for ASEP it reduces to the single line $\rho_l < \rho_r$, $\rho_l + \rho_r = 1$.

Remark. When the bulk *and* boundary dynamics satisfy detailed balance conditions as in Ref. 12, the correct order for $D^N(\mu^N)$ is even $O(1/N)$, as shown in that paper. This is because the positive entropy production at the boundaries may be interpreted as a current, and the mean current is of order $1/N$ by Fick's law. This corresponds to $p = q$ in (18). However the boundary terms in (18) are still of order $O(1)$ in that case. In order to obtain boundary terms that can be interpreted as current, one should use (as in Ref. 12) the exact entropy production

$$\int f_t(\eta)\mathcal{L}_{\text{op}}^N \ln f_t(\eta)\, d\nu^N$$

instead of its upper bound

$$2\int \sqrt{f_t(\eta)}\mathcal{L}_{\text{op}}^N \sqrt{f_t(\eta)}\, d\nu^N$$

Assume that, in the setting of Ref. 12, weakly asymmetric jump rates $N^{-1}c'(x, y, \eta)$ are added, such that $c'(x, y, \eta)$ satisfies (a) of the stationarity condition w.r.t the same Gibbs measures for which $c(x, y, \eta)$ satisfy detailed balance (this is what happens in WASEP). Repeating the arguments of Subsec. 4.2 for c', we see that the contribution of the weakly asymmetric part *without* the N^{-1}

factor is already only $O(1)$. Thus the overall $O(1/N)$ bound of Ref. 12 for the Dirichlet form is maintained; this is why we claim that the results of Ref. 12 extend to this setting.

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